I. INTRODUCTION

Direction-finding antenna arrays possess certain parameters which determine the array response, and whose accurate knowledge is paramount to a successful operation of the array. Consider for example a planar array of $M$ elements. A typical set of parameters includes, for each element of the array (say the $m$th), the following: maximum gain $g_m$, electrical phase angle $\alpha_m$ relative to a specified reference point, displacements $(d_{x,m}, d_{y,m})$ relative to the reference point and a reference direction, and in case of directional antennas, the boresight angle $\beta_m$ (the angle of maximum gain) with respect to the reference direction.

After the array is manufactured and installed in place, the parameters are known only up to a certain tolerance, which is typically not sufficient for on-line operation. Therefore, a calibration procedure is often necessary, during which the parameters are estimated to the desired accuracy. Furthermore, the parameters often vary in time due to various environmental effects, so periodic recalibration becomes necessary. Calibration is typically performed by placing a strong transmitter at a known location, and measuring the signals received by the array elements. This is repeated for various transmitter locations, in the interest of spatial diversity. Then some estimation algorithm is used to find the unknown parameters. We remark that sometimes the estimation phase is skipped altogether, and a calibration table is used instead. Here, however, we are interested in the case of parametric calibration.

Since the calibration procedure is likely to be time consuming and expensive, it is desirable to minimize the number of transmission points, as well as the number of snapshots taken at each point, while achieving a desired accuracy of the calibration parameters. We study the requirements from the calibration procedure so as to meet the desired accuracy of the direction-of-arrival (DOA) estimates during on-line operation. We consider the following three performance measures.

1) The first measure is the on-line DOA estimation error covariance due to noise. This depends on the signal-to-noise ratio (SNR) in the on-line scenario, the number of snapshots, the number of sources, the angular separation of the sources, and the DOA estimation algorithm.

2) The error variance of the calibration parameters in the calibration procedure is the second measure. This depends on the SNR during calibration, the number of transmission points and their locations, and the number of snapshots taken at each transmission point.

3) The third measure is the sensitivity matrix of the DOA errors with respect to the errors in the calibration parameters.
Based on these three measures, we propose a criterion to decide whether the calibration accuracy is satisfactory. This criterion can be used for choosing the calibration setup so as to meet the requirements with a minimum number of transmission points and snapshots.

Multipath is a prevalent and difficult problem in array calibration. Multipath introduces bias into the measurements, hence bias to the estimates. It is not uncommon that the error in the estimated parameters due to multipath is larger than the error due to noise. It is therefore important to analyze the bias in the estimates due to multipath. This analysis is also carried out in this work.

There have been a number of important contributions to the problem of array calibration in recent years, see [1-10] for examples. In particular, considerable attention has been given to self-calibration algorithms. The fact remains, however, that despite these efforts, many practical systems still rely on preliminary off-line calibration. The problem addressed here, namely, the design of off-line procedures to meet on-line performance requirements, has not been systematically treated to our knowledge.

The plan of the paper is as follows. In Section II we introduce some notations and definitions. In Section III we derive the three accuracy measures pertinent to our analysis. In Section IV we introduce the proposed calibration criterion. In Section V we analyze the effect of multipath on the calibration accuracy. Section VI shows some examples, and Section VII concludes the paper.

II. PRELIMINARIES

Each element of the array is characterized by $K$ parameters. We denote the $k$th parameter of the $m$th antenna by $\psi_{k,m}$. The $K$-vector of parameters of the $m$th antenna is denoted by $\psi_{m}$, and the $M$ vector consisting of the $k$th parameter of each antenna is denoted by $\psi_{k}$. The $KM$ vector of all parameters of all antennas is denoted by $\psi$. To be specific, we assume that $K = 5$, and $\psi_{m} = (g_{m}, \alpha_{m}, \delta_{x,m}, \delta_{y,m}, \beta_{m})$, as defined in Section I. The analysis in the paper is general, however, and can be used for more complicated models, such as antenna coupling effects.

Each antenna is assumed to have a known gain pattern $g(\cdot)$. The displacements $(\delta_{x,m}, \delta_{y,m})$ are measured in units of $\lambda/2\pi$, where $\lambda$ is the wavelength. Then the response of the $m$th antenna to a unit amplitude plane wave coming from direction $\theta$ is given by

$$a_{m}(\theta, \psi_{m}) = g_{m}g(\theta - \beta_{m}) \times \exp\{j(d_{x,m} \sin \theta + d_{y,m} \cos \theta + \alpha_{m})\}. \tag{1}$$

The vector of $\{a_{m}(\theta, \psi_{m}), 1 \leq m \leq M\}$ (the so-called steering vector) is denoted by $a(\theta, \psi)$. When there are multiple sources, say $Q$ in number, we denote the corresponding $Q$ vector of DOAs by $\theta$. The $M \times Q$ matrix whose $q$th column is $a(\theta_{q}, \psi)$ is denoted by $A(\theta, \psi)$.

During on-line operation, the signals received at the $n$th snapshot are the components of the $M$ vector

$$x_{n} = A(\theta, \psi)s_{n} + w_{n}, \quad 1 \leq n \leq N \tag{2}$$

where:

1) $s_{n}$ is the $Q$ vector of complex amplitudes of the sources at the $n$th snapshot. We assume that the amplitudes are deterministic quantities, and that the limiting matrix

$$S = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} s_{n} s_{n}^{H} \tag{3}$$

exists and is nonsingular.

2) $w_{n}$ is the noise $M$ vector at the $n$th snapshot. The noise is assumed Gaussian, temporally and spatially uncorrelated, with covariance matrix $\sigma_{w}^{2}I$.

Calibration is assumed to be performed using $L$ transmitter locations successively (not simultaneously), and $N_{t}$ snapshots are assumed to be taken from the $l$th transmitter. The angle of the $l$th transmitter is denoted by $\theta_{l}$. The measurement model during calibration is therefore

$$x_{l,n} = a(\theta_{l}, \psi)s_{l,n} + w_{l,n}, \quad 1 \leq l \leq L, \quad 1 \leq n \leq N_{t} \tag{4}$$

where:

1) $s_{l,n}$ is the amplitude transmitted from the $l$th location at the $n$th snapshot. We assume that the amplitudes are deterministic quantities, and that the limits

$$S_{l} = \lim_{N_{t} \to \infty} \frac{1}{N_{t}} \sum_{n=1}^{N_{t}} |s_{l,n}|^{2} \tag{5}$$

exist.

2) $w_{l,n}$ is the noise $M$ vector at the $n$th snapshot of the $l$th location. The noise is assumed Gaussian, temporally and spatially uncorrelated, with covariance matrix $\sigma_{w}^{2}I$.

For the parametric model (1), the minimum number of calibration points is $L = 3$, regardless of the number of elements. The reason is as follows. The effective number of independent complex equations for parameter estimation is $LM$, since the $N_{t}$ measurements at each location serve only to increase the SNR, not to add more information. This translates to $LM$ equations for absolute values and $LM$ equations for phases. The $2M$ parameters $g_{m}$ and $\beta_{m}$ can be estimated only from the absolute
values, while the $3M$ parameters $d_{x,m}, d_{y,m}, \alpha_m$ can be estimated only from the phases. Therefore $L = 3$ is necessary for the estimation problem to be well posed. We also remark that, in practice, one of the array antennas is usually taken as a reference and its parameters are assumed to be perfectly known (then the parameters of all other antennas are defined relative to the reference antenna). This changes the factor $M$ in the above calculation to $M - 1$, but does not affect the conclusion.

III. ACCURACY MEASURES

A. On-Line Accuracy

The accuracy of the on-line DOA estimates depends on the source parameters, the noise, and the algorithm used. To be specific, we assume that the algorithm is maximum likelihood (ML). Let us denote by $B(\theta, \psi)$ the $M \times Q$ matrix whose $(m,q)$ element is

$$b_{m,q}(\theta_q, \psi_m) = \frac{\partial b_{m,q}(\theta_q, \psi_m)}{\partial \theta_q}$$

(6)

and

$$P_A(\theta, \psi) = I - A(\theta, \psi) \times [A^H(\theta, \psi)A(\theta, \psi)]^{-1}A^H(\theta, \psi)$$

(7)

$$H(\theta, \psi) = B^H(\theta, \psi)P_A(\theta, \psi)B(\theta, \psi).$$

(8)

The asymptotic covariance of the ML DOA errors was derived in [7], and is given by (abbreviating the notations of $A$ and $H$, for convenience)

$$\text{cov}\{\hat{\theta}\} = \frac{\sigma^2}{2N} [\Re\{H \odot S^T\}]^{-1} \times [\Re\{H \odot [S + \sigma^2(A^HA)^{-1}]\}] [\Re\{H \odot S^T\}]^{-1}$$

(9)

where $\Re\{\cdot\}$ denotes the real part, and $\odot$ denotes the Hadamard product.

For the sake of later comparisons, we derive two special cases of (9). In both cases we assume omnidirectional antennas, so the gain pattern $g(.)$ is identically 1.

1) Case of a single source. In this case we get, after a simple derivation, the well-known expression

$$\text{cov}\{\hat{\theta}\} = \frac{1}{2N \cdot \text{SNR} \cdot J(\theta)} \left(1 + \frac{1}{\text{SNR} \sum_{m=1}^{M} \alpha_m^2}\right)$$

(10)

where $J(\theta)$ is the moment-of-inertia of the array around an axis parallel to the source direction, defined by

$$J(\theta) = \sum_{m=1}^{M} \sigma_m^2 (d_{x,m} \cos \theta - d_{y,m} \sin \theta)^2$$

$$- \frac{\left[\sum_{m=1}^{M} g_m^2 (d_{x,m} \cos \theta - d_{y,m} \sin \theta)\right]^2}{\sum_{m=1}^{M} g_m^2}.$$ 

(11)

2) Case of two closely spaced sources, at angles $\theta \pm 0.5 \Delta \theta$. In this case we have

$$A(\theta, \psi) = [a(\theta + 0.5 \Delta \theta, \psi), a(\theta - 0.5 \Delta \theta, \psi)]$$

$$\approx [a(\theta, \psi) + 0.5b(\theta, \psi)\Delta \theta, a(\theta, \psi) - 0.5b(\theta, \psi)\Delta \theta]$$

(12)

$$B(\theta, \psi) = [b(\theta + 0.5 \Delta \theta, \psi), b(\theta - 0.5 \Delta \theta, \psi)]$$

$$\approx [b(\theta, \psi) + 0.5c(\theta, \psi)\Delta \theta, b(\theta, \psi) - 0.5c(\theta, \psi)\Delta \theta]$$

(13)

where $c(\theta, \psi)$ is the $M$ vector whose components are the second partial derivatives of $a_m(\theta, \psi, \psi_m)$ with respect to $\theta$.

The matrix on the right side of (12) spans the same column space as the matrix

$$A(\theta, \psi) \frac{\Delta}{\alpha} [a(\theta, \psi), b(\theta, \psi)].$$

(14)

Therefore,

$$P_A(\theta, \psi)B(\theta, \psi) \approx P_A(\theta, \psi)B(\theta, \psi)$$

$$\approx 0.5 P_A(\theta, \psi)c(\theta, \psi)[1, -1] \Delta \theta$$

(15)

$$H(\theta, \psi) \approx 0.25(\Delta \theta)^2 \mathcal{C}H(\theta, \psi)$$

$$\times [P_A(\theta, \psi)c(\theta, \psi) [1, -1]$$

$$[1, 1]$$

(16)

$$H \odot S^T \approx 0.25(\Delta \theta)^2 \mathcal{C}H(\theta, \psi)$$

$$\times [P_A(\theta, \psi)c(\theta, \psi)S^T$$

(17)

where $S^T$ is obtained from $S^T$ by inverting the polarities of its off-diagonal elements. The matrix $(A^HA)^{-1}$ is approximately given by

$$(A^HA)^{-1} \approx \frac{1}{(\Delta \theta)^2 J(\theta)} [1, -1] H \odot [S + \sigma^2(A^HA)^{-1}]^T \approx 0.25c^H(\theta, \psi)P_A(\theta, \psi)$$

$$\times c(\theta, \psi) [\Delta \theta]^2 S^T + \frac{\sigma^2}{J(\theta)} I_2$$

(19)
where \( I_2 \) is the \( 2 \times 2 \) matrix whose elements are all 1. Finally,

\[
\text{cov}\{\hat{\theta}\} \approx \frac{2\sigma^2}{N(\Delta \theta)^2} \psi^H(\theta, \psi) P_A(\theta, \psi) \psi(\theta, \psi) \text{[} R[S^T] \text{]}^{-1} \times \left[ (\Delta \theta)^2 S^T + \frac{\sigma^2}{f(\theta)} I_2 \right] \text{[} R[S^T] \text{]}^{-1}.
\]

(20)

The dependence of \( \text{cov}\{\hat{\theta}\} \) on \( \Delta \theta \) for small values \( \Delta \theta \) can be roughly divided into two regions. In the region \( |\Delta \theta| \ll \sigma/(|J(\theta)|/2||S||) \), \( \text{cov}\{\hat{\theta}\} \) is approximately proportional to \( (\Delta \theta)^{-2} \), while in the region \( |\Delta \theta| \gg \sigma/(|J(\theta)|/2||S||) \) it is approximately proportional to \( (\Delta \theta)^{-2} \). Of course, part or whole of the latter region is outside the range of the first-order approximations (12) and (13), so (20) will not hold there.

B. Sensitivity of DOA Errors to Array Parameters

The sensitivity of the DOA errors to the array parameters depends on the source parameters, and on the algorithm used for DOA estimation. In [2], sensitivity analysis was performed for the ML algorithm. It has been shown that the sensitivity matrix \( S \) is given by

\[
S = \left( \frac{\partial^2 L(\theta, \psi)}{\partial \theta^2} \right)^{-1} \left( \frac{\partial^2 L(\theta, \psi)}{\partial \theta \partial \psi} \right)
\]

(21)

where the two matrices on the right side are, respectively, the matrix of second partial derivatives of the likelihood function \( L(\theta, \psi) \) with respect to the components of \( \theta \) and the matrix of mixed partial derivatives of the likelihood function with respect to the components of \( \theta \) and \( \psi \). The likelihood function is given by

\[
L(\theta, \psi) = \text{tr}\{P(\theta, \psi)\hat{R}\}
\]

(22)

where \( \hat{R} \) is the sample covariance matrix of the measurements. The derivative matrices in (21) are to be evaluated at the true \( \theta \) and \( \psi \), and substituting the true covariance

\[
R = A(\theta, \psi) S A^H(\theta, \psi) + \sigma^2
\]

(23)

for \( \hat{R} \).

The second mixed partial derivative of \( L(\theta, \psi) \) with respect to arbitrary scalar variables \( x \) and \( y \) (which can be any of the components of \( \theta \) or \( \psi \)) can be shown, by a tedious but straightforward calculation, to be

\[
\frac{\partial^2 L(\theta, \psi)}{\partial x \partial y} = 2\Re \left\{ \text{tr} \left\{ \frac{\partial A^H}{\partial y} P \frac{\partial A}{\partial x} S \right\} \right\}.
\]

(24)

To continue, note the following properties of the matrix \( A(\theta, \psi) \).

1) The derivative of \( A \) with respect to \( \theta_q \) is an \( M \times Q \) matrix whose only non-zero column is the \( q \)th one.

2) The derivative of \( A \) with respect to \( \psi_{m,k} \) is an \( M \times Q \) matrix whose only non-zero row is the \( m \)th one.

Consider (24) for \( x = \theta_q \) and \( y = \theta_r \). Then we have, by the first property of \( A \),

\[
\text{tr} \left\{ \frac{\partial A^H}{\partial \theta_r} P \frac{\partial A}{\partial \theta_q} S \right\} = \sum_{i_1} \sum_{i_2} \sum_{i_4} \sum_{i_5} \left( \frac{\partial A^H}{\partial \theta_r} \right)_{i_1,i_2} \times P_{i_2,i_4} \left( \frac{\partial A}{\partial \theta_q} \right)_{i_4,i_5} \left( S_{i_5,i_1} \right)
\]

(25)

Therefore, the \( Q \times Q \) matrix of second partial derivatives with respect to \( \theta \) is

\[
\frac{\partial^2 L(\theta, \psi)}{\partial \theta^2} = 2\Re \{(B^H P^\perp B) \odot S^T\}.
\]

(26)

Next consider (24) for \( x = \theta_q \) and \( y = \psi_{m,k} \). Let \( E_k \) denote the \( M \times Q \) matrix whose \( m \)th row is the derivative of the \( m \)th row of \( A \) with respect to \( \psi_{m,k} \). Then we have, by the second property of \( A \),

\[
\text{tr} \left\{ \frac{\partial A^H}{\partial \psi_{m,k}} P \frac{\partial A}{\partial \theta_q} S \right\} = \sum_{i_1} \sum_{i_2} (E_k)_{i_1,m} \sum_{i_4} \left( \frac{\partial A^H}{\partial \psi_{m,k}} \right)_{i_1,i_2} \times P_{i_2,i_4} \left( \frac{\partial A}{\partial \theta_q} \right)_{i_4,i_5} \left( S_{i_5,i_1} \right)
\]

(29)

Therefore, the \( Q \times M \) matrix of the mixed partial derivatives with respect to \( \theta \) and \( \psi_{k} \), is

\[
\frac{\partial^2 L(\theta, \psi)}{\partial \theta \partial \psi_{k}} = 2\Re \{(B^H P^\perp B) \odot S^T\}.
\]

(30)

In summary, the sensitivity matrix of \( \theta \) with respect to \( \psi_{k} \), is

\[
S_k = [\Re \{(B^H P^\perp B) \odot S^T\}]^{-1} [\Re \{(S E_k^H) \odot (P^\perp B)^T\}]
\]

(31)

and the complete sensitivity matrix is

\[
S = [S_1, S_2, \ldots, S_k].
\]

(32)

Consider the case of two closely spaced sources at angles \( \theta \pm 0.5\Delta \theta \), as before. Using the results (15) and
in (31), we get
\[ S_k \approx \frac{2}{(\Delta \theta) \mathbf{e}^H(\theta, \psi) \mathbf{P}_A(\theta, \psi) \mathbf{e}^T(\theta, \psi) [\mathbb{R}\{\mathbf{S}_k\}]^{-1}} \times [\mathbb{R}\{(\mathbf{S}_k)^H \mathbf{P}_A(\theta, \psi) \mathbf{e}^T(\theta, \psi) \mathbf{e}^T(\theta, \psi)^T]\}]. \]

The fact that $\mathbf{S}$ is approximately proportional to the first power of $(\Delta \theta)^{-1}$, while $\text{cov}\{\mathbf{\theta}\}$ is proportional to its 4th or 2nd power [cf. (20) and the discussion that follows it] has interesting implications, as we discuss later.

C. Off-Line Accuracy

Consider a general nonlinear model of the form
\[ \mathbf{z} = \mathbf{y}(\mathbf{p}_0) + \mathbf{e} \] (34)
where $\mathbf{p}_0$ is the parameter vector of interest, and $\mathbf{e}$ is an error vector, deterministic or random, whose norm is assumed small with respect to that of $\mathbf{y}(\mathbf{p}_0)$. Suppose we estimate the parameter vector using nonlinear weighted least-squares, using the weight matrix $\mathbf{W}$. The first-order approximation of this estimate is well known to be
\[ \mathbf{\hat{p}} \approx \mathbf{p}_0 + \left[\Gamma^T(\mathbf{p}_0)\mathbf{W}(\mathbf{p}_0)\right]^{-1}\Gamma^T(\mathbf{p}_0)\mathbf{W}\mathbf{e} \] (35)
where
\[ \Gamma(\mathbf{p}) = \frac{\partial \mathbf{y}(\mathbf{p})}{\partial \mathbf{p}}. \] (36)

If $\mathbf{e}$ is random with zero mean and covariance matrix $\mathbf{W}^{-1}$, then the covariance of $\mathbf{\hat{p}}$ is
\[ \text{cov}\{\mathbf{\hat{p}}\} \approx \left[\Gamma^T(\mathbf{p}_0)\mathbf{W}(\mathbf{p}_0)\right]^{-1}. \] (37)

When the measurements are complex, we need to modify the above expressions as follows. First we split (34) into its real and imaginary parts and rewrite it as
\[ \begin{bmatrix} \mathbf{z}_r \\ \mathbf{z}_i \end{bmatrix} = \begin{bmatrix} \gamma_r(\mathbf{p}_0) \\ \gamma_i(\mathbf{p}_0) \end{bmatrix} + \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_i \end{bmatrix} \] (38)
Assume also that we use a block diagonal weight matrix whose two blocks are both $\mathbf{W}$. Then we get from (35),
\[ \mathbf{\hat{p}} \approx \mathbf{p}_0 + \left[\Gamma^T(\mathbf{p}_0)\mathbf{W}_r(\mathbf{p}_0) + \Gamma^T(\mathbf{p}_0)\mathbf{W}_i(\mathbf{p}_0)\right]^{-1} \times \left[\Gamma^T(\mathbf{p}_0)\mathbf{W}_r\mathbf{e}_r + \Gamma^T(\mathbf{p}_0)\mathbf{W}_i\mathbf{e}_i\right]. \] (39)
If $\mathbf{e}$ is a complex random vector with zero mean, uncorrelated real and imaginary parts, each having covariance matrix $\mathbf{W}^{-1}$, then the covariance of $\mathbf{\hat{p}}$ is
\[ \text{cov}\{\mathbf{\hat{p}}\} \approx \left[\Gamma^T(\mathbf{p}_0)\mathbf{W}_r(\mathbf{p}_0) + \Gamma^T(\mathbf{p}_0)\mathbf{W}_i(\mathbf{p}_0)\right]^{-1}. \] (40)

Equations (39) and (40) can also be written in terms of the complex entities:
\[ \mathbf{\hat{p}} \approx \mathbf{p}_0 + \left[\mathbb{R}\{\Gamma^H(\mathbf{p}_0)\mathbf{W}(\mathbf{p}_0)\}\right]^{-1}\mathbb{R}\{\Gamma^H(\mathbf{p}_0)\mathbf{W}\mathbf{e}\} \] (41)

In off-line calibration we have two options. We can rely on the knowledge of the amplitudes $\mathbf{s}_{1:n}$ and estimate only the calibration parameters $\mathbf{\psi}$. In this case $\mathbf{p}_0 = \mathbf{\psi}$. Or, we can ignore our prior knowledge of the amplitudes and estimate them as well. In this case $\mathbf{p}_0$ consists of the $\mathbf{K}\mathbf{M}$ components of $\mathbf{\psi}$ and the $\sum_{l=1}^L N_l$ amplitudes. The latter possibility is not necessarily much more complicated than the former, since the measurement model (4) is linear in the amplitudes. In any case our interest here is in the accuracy of $\mathbf{\hat{\psi}}$, so we compute it now for the two cases.

1) The amplitudes are known and used in the calibration. Then the measurement vector $\mathbf{y}$ is obtained by stacking the real parts and imaginary parts of the following vectors in one column:
\[ \gamma_{l,n} = \mathbf{a}(\theta_l, \psi)\mathbf{s}_{l,n}, \quad 1 \leq l \leq L, \quad 1 \leq n \leq N_l. \] (43)
Correspondingly, the partial derivative matrix $\mathbf{F}$ is obtained by stacking the following matrices in a column block-matrix:
\[ \Gamma_{l,n} = F(\theta_l, \psi)\mathbf{s}_{l,n} \] (44)
where
\[ F(\theta_l, \psi) = \frac{\partial \mathbf{a}(\theta_l, \psi)}{\partial \psi}. \] (45)
The matrix $F(\theta_l, \psi)$ is $\mathbf{M} \times \mathbf{K}\mathbf{M}$. Note that the entries of this matrix are basically the same as the ones in the matrices $\mathbf{E}_k$ defined previously. Specifically, $F(\theta_l, \psi)$ consists of $\mathbf{K}$ blocks of dimensions $\mathbf{M} \times \mathbf{M}$ in a row. The $k$th block is diagonal (since $a_m$ depends only on $\psi_{nm}$) and its diagonal elements are equal to those of the $\mathbf{M} \times 1$ matrix $\mathbf{E}_k$.

The covariance of $\mathbf{\hat{\psi}}$ is given by
\[ \text{cov}\{\mathbf{\hat{\psi}}\} \approx \mathbb{R}\left\{\sum_{l=1}^L \frac{1}{\sigma_l^2} \left(\sum_{n=1}^{N_l} |\mathbf{s}_{l,n}|^2\right) F^H(\theta_l, \psi)F(\theta_l, \psi)\right\}^{-1}. \] (46)
Assuming that $N_l$ is large and using (5), we can approximate (46) by
\[ \text{cov}\{\mathbf{\hat{\psi}}\} \approx \mathbb{R}\left\{\frac{1}{\sigma_l^2} \sum_{l=1}^L N_l |\mathbf{s}_{l}^H F(\theta_l, \psi)F(\theta_l, \psi)|^{-1}\right\}^{-1}. \] (47)
It is convenient to define a scaled signal power by
\[ S_{\infty,l} = S_l \cdot \frac{N_l}{N} \cdot \frac{\sigma_l^2}{\sigma_l^2} \] (48)
write (47) as
\[
\text{cov}\{\hat{\psi}\} \approx \frac{\sigma^2}{N} \left[ \Re \left\{ \sum_{l=1}^{L} S_{\text{sc}} F^H(\bar{\theta}_l, \psi)F(\bar{\theta}_l, \psi) \right\} \right]^{-1}.
\] (49)

2) The amplitudes are estimated during calibration. The measurement vector \(y\) is as in the previous case. However, the matrix \(\Gamma\) is obtained by stacking the following matrices in a column block-matrix:
\[
\Gamma_{Gn} = [F(\bar{\theta}_1, \psi)S_{l,n}, 0, \ldots, a(\bar{\theta}_1, \psi), \ldots, 0]
\] (the width of this matrix is \(KM + \sum_{l=1}^{L} N_l\))

\[
\Gamma_{Gn}^H \Gamma_{Gn} =
\begin{bmatrix}
|s_{l,n}|^2 F^H(\bar{\theta}_1, \psi)F(\bar{\theta}_1, \psi) & 0 & \cdots & s_{l,n}^H F^H(\bar{\theta}_1, \psi)a(\bar{\theta}_1, \psi) & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
s_{l,n} F^H(\bar{\theta}_1, \psi)F(\bar{\theta}_1, \psi) & 0 & \cdots & a^H(\bar{\theta}_1, \psi)a(\bar{\theta}_1, \psi) & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\] (50)

Now the covariance of \(\hat{\psi}\) is given by
\[
\text{cov}\{\hat{\psi}\} \approx \left[ \Re \left\{ \sum_{l=1}^{L} s_{l,n}^2 F^H(\bar{\theta}_1, \psi)F(\bar{\theta}_1, \psi) \right\} \right]^{-1}.
\] (52)

As before, we can approximate this result by
\[
\text{cov}\{\hat{\psi}\} \approx \frac{\sigma^2}{N} \left[ \Re \left\{ \sum_{l=1}^{L} S_{\text{sc}} F^H(\bar{\theta}_1, \psi)P_A^\dagger(\bar{\theta}_1, \psi)F(\bar{\theta}_1, \psi) \right\} \right]^{-1}.
\] (53)

IV. CRITERION FOR OFF-LINE CALIBRATION

When the calibration procedure yields an estimate \(\hat{\psi}\), the resulting error in the on-line DOA estimate is
\[
\hat{\theta} - \theta = S(\hat{\psi} - \psi).
\] (54)

The quantity \(\mu(\text{tr}\{\text{cov}\{\hat{\psi}\}\})^{1/2}\) can be regarded as a practical upper bound on \(\|\hat{\psi} - \psi\|\). The factor \(\mu\) is determined by the desired probability level. For example, if we assume that \(\psi\) is approximately Gaussian distributed, then \(\mu = 3\) corresponds to probability 99.7%. Therefore, the quantity \(\mu \||S||\text{tr}\{\text{cov}\{\hat{\psi}\}\}\}^{1/2}\) is a practical upper bound on \(\|\hat{\theta} - \theta\|\), where \(\||S||\) is the largest singular value of \(S\).

Suppose we want this to be a fraction, say \(\kappa\), of the practical upper bound on \(\|\hat{\theta} - \theta\|\) due to noise in the on-line scenario, which is \(\mu(\text{tr}\{\text{cov}\{\hat{\psi}\}\})^{1/2}\). Then, the requirement is
\[
[\text{tr}\{\text{cov}\{\hat{\psi}\}\}]^{1/2} \leq \kappa ||S||^{-1}[\text{tr}\{\text{cov}\{\hat{\psi}\}\}]^{1/2}.
\] (55)

Consider again the case of two closely spaced sources. In (33), the matrix on the right of the operator \(\odot\) has two identical rows, except for sign. The matrix \(E_k\) is rank two, but its columns are almost identical, since so are the columns of \(A\). Let us further assume that \(S\) is diagonal, that is, the two sources are uncorrelated. In this case \(S\) is also real. It is seen that

\[
\text{cov}\{\hat{\psi}\} \approx \frac{\sigma^2}{N} \left[ \Re \left\{ \sum_{l=1}^{L} S_{\text{sc}} F^H(\bar{\theta}_1, \psi)F(\bar{\theta}_1, \psi) \right\} \right]^{-1}.
\] (52)

As before, we can approximate this result by
\[
\text{cov}\{\hat{\psi}\} \approx \frac{\sigma^2}{N} \left[ \Re \left\{ \sum_{l=1}^{L} S_{\text{sc}} F^H(\bar{\theta}_1, \psi)P_A^\dagger(\bar{\theta}_1, \psi)F(\bar{\theta}_1, \psi) \right\} \right]^{-1}.
\] (53)

This causes cancellation of \(S\) in (33), and we get the approximation
\[
S_k \approx \frac{2}{(\Delta \theta)e^H(\theta, \psi)P_A^\dagger(\theta, \psi)e(\theta, \psi)} \\
\times [\Re \{E_k^H \odot \{P_A^\dagger(\theta, \psi)e(\theta, \psi)(1, -1)\}^T\}].
\] (56)

Therefore \(S\) is "almost rank one", in the sense that one of its singular values is nearly zero. Its other singular value, which determines \(\|S\|\), is approximately \(2^{1/2}\) times the norm of either of its two rows, that is
\[
\|S\| \approx \frac{2^{3/2}}{(\Delta \theta)e^H(\theta, \psi)P_A^\dagger} \left[ \sum_{k=1}^{K} \left\| \Re \{E_k^H \odot (P_A^\dagger e)^T\} \right\|^2 \right]^{1/2}.
\] (57)

Finally, substitution of (57) and (20) in (55) gives
\[
[\text{tr}\{\text{cov}\{\hat{\psi}\}\}]^{1/2} \leq \frac{\kappa \sigma}{2 \Delta \theta} \left[ e^H(\theta, \psi) \text{tr}(S^{-1}) + \sigma^2 J^{-1}(\theta) \text{tr}(S^{-1} \odot S^{-1}) \right]^{1/2}
\] (58)

The resulting expression on the right side of (58) has some interesting implications. The dependence on the inverse of \(\Delta \theta\) for small \(\Delta \theta\) implies that the accuracy requirements on \(\hat{\psi}\) become less stringent as the separation between the sources decreases. This
largely depends on the correlation between \( s_{i,n}^{(m)} \) and the amplitude \( s_{i,n} \) received from the calibration transmitter at the \( i \)th point. The best case, from calibration accuracy viewpoint, is when they are uncorrelated, since then averaging of the error \( e_{i,n} \) over \( n \) is the most effective. Unfortunately, this is rarely the case, since multipath reflections originate from the same transmitter that generates the direct signal. The worst case is when \( s_{i,n}^{(m)} \) is fully correlated with \( s_{i,n} \), meaning that

\[
e_{i,n} = A(\theta^{(m)}, \psi)\eta_i s_{i,n}.
\]

where \( \eta_i \) is a constant vector (independent of \( n \)).

Here we are interested only in the worst case (i.e., fully correlated) multipath, since calibration has to meet the accuracy requirements under worst case conditions. The multipath error vector is then given by

\[
e_{i,n} = A(\theta^{(m)}, \psi)\eta_i s_{i,n}.
\]

As in Section III C, we examine two cases.

1) The amplitudes are known and used in the calibration. Then we get from (44) and (63)

\[
\Gamma^H W e = \frac{\sigma_i}{\sigma_i^2} \left( \sum_{l=1}^{L} |s_{i,n}|^2 \right) F^H(\vartheta, \psi) A(\theta^{(m)}, \psi) \eta_i.
\]

Using (5) and the scaled signal power definition (48), we can express (64) as

\[
\Gamma^H W e = \frac{N \sigma_i}{\sigma_i^2} \sum_{l=1}^{L} S_{\vartheta, l} F^H(\vartheta, \psi) A(\theta^{(m)}, \psi) \eta_i.
\]

Finally, substituting (65) and (49) in (41) we get the following expression for the parameter error due to fully correlated multipath:

\[
\Delta \hat{\psi} \approx \left[ \mathbb{R} \left\{ \sum_{l=1}^{L} S_{\vartheta, l} F^H(\vartheta, \psi) F(\vartheta, \psi) \right\} \right]^{-1} \times \mathbb{R} \left\{ \sum_{l=1}^{L} S_{\vartheta, l} F^H(\vartheta, \psi) A(\theta^{(m)}, \psi) \eta_i \right\}.
\]

2) The amplitudes are estimated during calibration. Using the same method of derivation as above and as in Section III C, we get for this case

\[
\Delta \hat{\psi} \approx \left[ \mathbb{R} \left\{ \sum_{l=1}^{L} S_{\vartheta, l} F^H(\vartheta, \psi) P_A^*(\vartheta, \psi) F(\vartheta, \psi) \right\} \right]^{-1} \times \mathbb{R} \left\{ \sum_{l=1}^{L} S_{\vartheta, l} F^H(\vartheta, \psi) P_A^*(\vartheta, \psi) A(\theta^{(m)}, \psi) \eta_i \right\}.
\]

An important conclusion from (66), (67) is that we cannot rely on measurement averaging to combat a
fully correlated multipath. The only recourse in such an event is to rely on transmitter location diversity, that is, to increase $L$ in a hope that the summation over $l$ in the right matrix on the right side of the formula will provide sufficient averaging to reduce $\Delta \hat{\phi}$. This is likely to succeed if multipath is due to objects located near the transmitter, so each transmission point has its own characteristic multipath reflections. If the objects are located near the array, the averaging operation in (66) or (67) is not likely to be effective. A reflector near the array appears to the array as a source of a fixed direction, and needs to be accounted for by different means, for example, by adding it to the calibration model and estimating its parameters. The details of such a procedure are outside the scope of this work.

VI. NUMERICAL EXAMPLES

In this section we illustrate the calibration criterion developed here by a few examples, and attempt to draw some general conclusions.

The first example is that of a circular uniform array of $M$ sensors, whose radius is chosen such that adjacent elements are spaced by $\lambda/2$. The nominal gains $g_m$ are all 1, and the nominal electrical angles $\alpha_m$ are all 0. There are two sources of equal power $S = 1$, located at angles $\pm 0.5 \Delta \theta$ with respect to boresight. We compute the quantity $||S||^{-1}[\text{tr}(\text{cov}(\hat{\theta}))]^{1/2}$ appearing on the right side of (55) in three different ways: first exactly, then using the approximation on the right side of (59), and finally the lower bound given on the right side of (60).

Fig. 1 shows the results as a function of $\Delta \theta$, on a log-log scale, for $M = 6$, $0.2^\circ \leq \Delta \theta \leq 40^\circ$ and $\sigma^2 = 1$. The solid line shows the exact results, the dashed line shows the small angle approximation, and the dot-dashed line shows the lower bound of the small angle approximation. As we see, the small angle approximation is in excellent agreement with the exact result up to $\Delta \theta = 10^\circ$. We also see that the log-log graph is a straight line for small angles, having a slope of 1 decade/decade. This confirms the $\Delta \theta^{-1}$ dependence of the criterion. The lower bound becomes almost tight at $\Delta \theta = 20^\circ$. This justifies it use as a practical measure against which $[\text{tr}(\text{cov}(\hat{\theta}))]^{1/2}$ is to be tested.

Figs. 2 and 3 show similar results for $\sigma^2 = 0.1$ and $\sigma^2 = 10$, respectively. As we see, changing the value
of $\sigma^2$ changes the range of $\Delta \theta^{-1}$ dependence, but otherwise the conclusions hold as before.

Next we tested a calibration scenario in which the $L$ calibration points are distributed circularly around the array, so the angle between neighboring points is $2\pi/L$. We let all scaled signal levels $S_{sc,l}$ be equal, and search for the smallest $L$ for which the criterion (60) is met, with $\kappa = 1$. We have found that $L$ is inversely proportional to $S_{sc,l}$. In particular, when $S_{sc,l} = 1$ we get $L = 78$ if the signal amplitudes are used in the calibration, and $L = 130$ if they are not. Practically this means that, if the calibration experiment is done under conditions similar to on-line operation (in SNR and number of measurements), we need an unreasonably large number of calibration points. On the other hand, if $S_{sc,l} = 10$ we get $L = 8$ or $L = 13$, which are very reasonable numbers. Working with $S_{sc,l} = 10$ or higher should not be difficult in most cases, since during calibration it is usually possible to get good SNR and a large number of measurements.

The second example is that of a sparse linear array of 5 sensors. The interelement spacings are $\{\lambda/2, 3\lambda/2, 5\lambda/2, \lambda\}$. The nominal gains $g_m$ are all 1, and the nominal electrical angles $\alpha_m$ are all 0. There are two sources of equal power $S = 1$, located at angles $\pm 0.5\Delta \theta$ with respect to boresight. We compute the quantity $||S||^{-1} [\text{tr} \{\text{cov} \{\tilde{\theta}\}\}]^{1/2}$ appearing on the right side of (55) as in the first example.

Fig. 4 shows the results as a function of $\Delta \theta$, on a log-log scale, for $0.4^\circ \leq \Delta \theta \leq 80^\circ$ and $\sigma^2 = 1$. The different line styles mean the same as in the first example. As we see, the small angle approximation is in excellent agreement with the exact result up to $\Delta \theta = 2^\circ$. At larger separation angles, the exact criterion behaves in an irregular manner and even becomes lower than the lower bound of the small angle approximation (so this “bound” is not really a bound in this case). Figs. 5 and 6 show similar results for $\sigma^2 = 0.1$ and $\sigma^2 = 10$, respectively. As we see, the decrease of the exact criterion below the lower bound of the small angle approximation is more pronounced at $\sigma^2 = 0.1$, and disappears at $\sigma^2 = 10$.

Next we tested a calibration scenario in which the $L$ calibration points are distributed uniformly in the range $[-85^\circ, 85^\circ]$. We let all scaled signal levels $S_{sc,l}$ be equal, and search for the smallest $L$ for which the criterion (60) is met. We have found that when $S_{sc,l} = 10$, the minimum number of calibration points is 19 if the signal amplitudes are used in the calibration, and 32 if they are not. When $S_{sc,l}$ is increased to 20, the
number of calibration points reduces to 8 or 14, and when $S_{sc,t}$ is increased to 40, the number of calibration points reduces to 3 or 5. In this case, therefore, the improvement in number of calibration points as a function of the SNR is better than linear.

The third example is that of a uniform linear array of $M$ sensors, spaced at intervals of $\lambda/2$. Here we tested the dependence of the number of calibration points $L$ on the array size $M$. We chose the scaled signal level so that with $M = 3$ and $L = 3$, the criterion is met exactly. Recall from Section II that $L = 3$ is the minimum number of calibration points for which the calibration problem is well defined. We denote the corresponding value of the scaled signal level by $S_{sc,0}$. We found that $S_{sc,0} = 17.48$ in the case of known amplitudes and $S_{sc,0} = 23.31$ in the case of unknown amplitudes. We then increased $M$ and, for each $M$, increased $L$ until the criterion was met. The $L$ transmitter locations are spaced uniformly in angle in the range $[-85^\circ, 85^\circ]$. We found that the following relationships describe the required $L$ as a function of $M$:

Known amplitudes:

$$L = \text{round}(2.5M - 4)$$ (68)

Unknown amplitudes:

$$L = \text{round}(3.8M - 8).$$ (69)

For other values of $S_{sc}$, we found the following experimental relationships:

Known amplitudes:

$$L = \max \left\{ \frac{S_{sc,0}}{S_{sc}} \cdot \text{round}(2.5M - 4), 3 \right\}$$ (70)

Unknown amplitudes:

$$L = \max \left\{ \frac{S_{sc,0}}{S_{sc}} \cdot \text{round}(3.8M - 8), 3 \right\}. $$ (71)

The last example illustrates the effect of fully correlated multipath on the calibration accuracy. We took again the circular array of the first example, and simulated multipath as follows. At each calibration point we have either no multipath or one multipath reflection, with probabilities 0.8 and 0.2, respectively. The multipath reflection, if present, has angular deviation from the transmitter that is chosen at random according to a Gaussian distribution with zero mean and standard deviation $1^\circ$. The relative amplitude of the reflection (i.e., the parameter $\eta$) chosen at random according to a complex circular Gaussian distribution with zero mean and standard deviation 0.2. We assume that the on-line parameters $\sigma^2$ and $N$ are 1 and 100, respectively. Under these assumptions, the threshold on $[\tr\{\cov\{\hat{\psi}\}\}]^{1/2}$ is 0.0623. Therefore, the value of $(\Delta\hat{\psi}^T \Delta\hat{\psi})^{1/2}$ should be a small fraction of this number for the multipath to be negligible. We found that about $L = 200$ calibration points are necessary in this case. Recall that, without multipath, only 8 calibration points were needed. The conclusion is that fully correlated multipath can be very costly in terms of number of calibration points needed to reduce its effect to an acceptable level.

VII. CONCLUSIONS

We developed a method of analysis of the accuracy needed in off-line calibration of sensor array to meet on-line performance requirements. We showed that the main parameters that need to be chosen for calibration are the number of transmitter locations (calibration points), their spatial distribution, and the scaled SNR during the calibration transmissions. We illustrated by some numerical examples the tradeoffs between these parameters. We have also analyzed the sensitivity of the calibration to multipath reflections, and showed that multipath can adversely affect the procedure to a great degree. Fully correlated multipath can only be combated by significantly increasing the number of calibration points.

APPENDIX. DETAILED CALCULATIONS

This Appendix contains some derivations needed for the approximations in the case of two closely spaced sources.

Let us write

$$a_m(\theta, \psi, \cdot) = g_m \exp\{j\phi(\theta, \psi, \cdot)\}. $$ (72)

Then,

$$b_m = \frac{\partial a_m(\theta, \psi, \cdot)}{\partial \theta} = j \frac{\partial \phi(\theta, \psi, \cdot)}{\partial \theta} a_m(\theta, \psi, \cdot) $$ (73)

$$c_m = \frac{\partial b_m(\theta, \psi, \cdot)}{\partial \theta} = \left[ - \left( \frac{\partial \phi(\theta, \psi, \cdot)}{\partial \theta} \right)^2 + j \frac{\partial^2 \phi(\theta, \psi, \cdot)}{\partial \theta^2} \right] a_m $$ (74)

$$e_{m,1} = \frac{\partial a_m(\theta, \psi, \cdot)}{\partial \psi_{1,m}} = (g_m)^{-1} a_m $$ (75)

$$e_{m,k} = \frac{\partial a_m(\theta, \psi, \cdot)}{\partial \psi_{k,m}} = j \frac{\partial \phi(\theta, \psi, \cdot)}{\partial \psi_{k,m}} a_m(\theta, \psi, \cdot), \quad 2 \leq k \leq K. $$ (76)
We need the following quantities:

\[ a^H a = \sum_{m=1}^{M} g_m^2 \]  
\[ a^H b = \sum_{m=1}^{M} g_m^2 \left( \frac{\partial \phi(\theta, \psi_m)}{\partial \theta} \right) \]  
\[ b^H b = \sum_{m=1}^{M} g_m^2 \left( \frac{\partial \phi(\theta, \psi_m)}{\partial \theta} \right)^2 \]  
\[ a^H c = \sum_{m=1}^{M} g_m^2 \left[ -\left( \frac{\partial \phi(\theta, \psi_m)}{\partial \theta} \right)^2 + \frac{\partial^2 \phi(\theta, \psi_m)}{\partial \theta^2} \right] \]  
\[ b^H c = \sum_{m=1}^{M} g_m^2 \left[ \frac{\partial^2 \phi(\theta, \psi_m)}{\partial \theta^2} \right] \times \frac{1}{\partial \theta} \phi(\theta, \psi_m) \]  

The vector \( P_A^+(\theta, \psi)c(\theta, \psi) \) is given by

\[ P_A^+(\theta, \psi)c(\theta, \psi) = c - \frac{1}{J(\theta) \sum_{m=1}^{M} g_m^2} \begin{bmatrix} a & b \\ b^H b & a^H b \end{bmatrix} \begin{bmatrix} b^H a \\ a^H a \end{bmatrix} \begin{bmatrix} b^H c \end{bmatrix} \]  

where

\[ J(\theta) = \left( \sum_{m=1}^{M} g_m^2 \right)^{-1} (a^H ab^H b - a^H bb^H a) \]  
\[ = \sum_{m=1}^{M} g_m^2 (d_{x,m} \cos \theta - d_{y,m} \sin \theta)^2 \]  
\[ - \sum_{m=1}^{M} g_m^2 (d_{x,m} \cos \theta - d_{y,m} \sin \theta)^2 \quad (83) \]  

is the moment-of-inertia of the array around an axis of direction \( \theta \).

REFERENCES


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